MTH 2310, FALL 2017

TEST 2 REVIEW, KEY

Some problems to work on in class today (most of these are even-numbered problems from the textbook):

(1) True/False: If True, justify your answer with a brief explanation. If False, give a counterexample or a brief explanation.

(a) If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are not in $\mathbb{R}^2$.
True. If it was in $\mathbb{R}^2$, the matrix would have more columns than rows and would thus have a free variable, so it could not be linearly independent.

(b) If $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, then $\mathbf{u}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$.
False. Let $\mathbf{w}$ be the zero vector, and $\mathbf{u}$ and $\mathbf{v}$ be linearly independent. Then the first part is okay because zero is always a linear combination, but the second part fails because $\mathbf{u}$ and $\mathbf{v}$ are not scalar multiples.

(c) If A is a 6×5 matrix, the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot map $\mathbb{R}^5$ onto $\mathbb{R}^6$.
True. When A is reduced it will have at most 5 pivots, and thus cannot have a pivot in every row, and so it cannot be onto.

(d) If A and B are 3×3 and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$, then $AB = \begin{bmatrix} A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3 \end{bmatrix}$.
False, should not have plusses (those should be columns of a matrix).

(e) If A is invertible, then the inverse of A^{-1} is A itself.
True. $(A^{-1})^{-1} = A$ is a property of inverses.

(f) Let A be a square matrix. If the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then the solution is unique for each $\mathbf{b}$.
True. Because of at least one solution we know it is onto, and by the IMT this means it is one-to-one.

(g) If A_1, A_2, B_1 and B_2 are $n \times n$ matrices, $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, then the product BA is defined but AB is not.
False, both are defined. The product BA is a $(n \times 2n)$ matrix times a $(2n \times n)$ matrix, and the product AB is a $(2n \times n)$ matrix times a $(n \times 2n)$ matrix. So BA will be an $n \times n$ matrix, and AB will be a $2n \times 2n$ matrix.

(h) The determinant of a triangular matrix is the product of the entries on the diagonal.
True. This follows from the calculation of the determinant because if you expand on the first row or column (the one which has all zero entries except for the diagonal entry) then you get the first entry times the determinant of a submatrix. Repeating this process over and over gives you the product of all the diagonal entries (this would

be a good situation to give an example for a 3×3 matrix or some such to show what you are explaining).

(i) If $\det A = 0$, then two rows or two columns are the same, or a row or column is zero. **False**, could have one row be a sum of two other rows. As long as one row (or column) is a linear combination of some of the other rows (or columns), the determinant will be zero.

(2) Determine the value(s) of a such that $\left\{ \begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} a \\ a+2 \end{bmatrix} \right\}$ is linearly independent.

Ans: The vectors will be linearly independent if $\det \begin{bmatrix} 1 & a \\ a & a+2 \end{bmatrix}$ is nonzero. This is when $a + 2 - a^2 \neq 0$, which factors to $-(a-2)(a+1) \neq 0$. Thus we will have linear independence as long as a is not 2 or -1.

(3) Let A be a 3×3 matrix with the property that the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^3$ onto $\mathbb{R}^3$. Explain (without using the IMT) why the transformation must be one-to-one.

Ans: If A is onto, it must have a pivot in every row. Since A is a 3×3, this means that it has 3 pivots. This means it has a pivot in every column as well, so it must be one-to-one. (If I had said nothing about the IMT, your answer would simply be "By the IMT, a square matrix that is onto is also one-to-one")

(4) Show that if the columns of B are linearly dependent, then so are the columns of AB. (Hint: First, explain why the columns of B being linearly dependent means the same thing as saying that there is a nonzero vector $\mathbf{v}$ such that $B\mathbf{v} = \mathbf{0}$. Now use this to answer the question.)

Ans: If the columns of B are linearly dependent, that means there are scalars $c_1, c_2, \ldots, c_n$ such that

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{0}$$

Where $\mathbf{b}_i$ are the columns of B and not all of the c_i are zero. This is the same as saying that $B \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ , c_n \end{bmatrix} = \mathbf{0}$. Multiplying both sides of that equation on the left by A gives

$$AB \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ , c_n \end{bmatrix} = \mathbf{0}$$

Since $AB\mathbf{x} = \mathbf{0}$ has a nontrivial solution (namely $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ , c_n \end{bmatrix}$) its columns must be linearly dependent.

(5) Find the inverse of $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$.

Ans: Put this into a online calculator to check your work. You should do it by hand using the algorithm where you augment it with the identity matrix and then row reduce.

(6) If L is an $n \times n$ matrix and the equation $L\mathbf{x} = \mathbf{0}$ has the trivial solution, do the columns of L span $\mathbb{R}^n$? Why or why not?

Ans: Not necessarily. If the equation has *only* the trivial solution, then it will span (by the IMT). But all such equations have the trivial solution, and some of them have more.

(7) Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$, where B and C are square. Prove that if A is invertible then B and C must be invertible. (Note: you cannot just invoke the Invertible matrix Theorem, you have to use block matrix multiplication at some point.)

Ans: A invertible means there is some block matrix $\begin{bmatrix} X & Y \\ W & Z \end{bmatrix}$ such that $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} X & Y \\ W & Z \end{bmatrix} = I$. This means that $BX = I$ so B must be invertible, etc.

(8) Calculate det $\begin{bmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix}$.

Ans: 9.

(9) Use row reduction to find det $\begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix}$.

Ans: 0.

(10) Let U be a square matrix such that $U^T U = I$. Show that either $\det U = 1$ or $\det U = -1$.

Ans: $\det(U^T U) = \det(I)$ implies $\det(U^T) \cdot \det(U) = 1$. But $\det U = \det U^T$, so they have to be ± 1.